

Note on the Signatures of Rough Paths in a Banach Space

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Abstract

We prove some results, which are used in [1], about weakly geometric rough paths that are well-known in finite dimensions, but need proof in the infinite dimensional setting.

1 Introduction

Lyons' rough path theory [9] investigates the meaning of the controlled differential equation

$$dy_t = V(y_t) dx_t \quad (1.1)$$

when x does not have a derivative. There are two natural classes of paths x for which the equation (1.1) is defined and the operation of integration satisfies the chain rule of classical calculus: the *geometric rough paths* and the *weakly geometric rough paths*. Let $p \geq 1$. The p -geometric rough paths (respectively p -weakly geometric rough paths) are the geometric rough paths (respectively weakly geometric rough paths) that have finite p -variation (see Definition 1.1 below). The *signature* of a rough path x on the interval $[0, T]$, defined formally as

$$S(x) = 1 + \int_0^T dx_{t_1} + \int_0^T \int_0^{t_2} dx_{t_1} \otimes dx_{t_2} + \dots,$$

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(see Definition 1.2 for precise definition) which arises naturally in rough path theory. In finite dimensions, a p -weakly geometric rough path is a p' -geometric rough path for all $p' > p$ (see [7]). As the map $x \rightarrow S(x)$ is continuous in the p -variation topology for all p , most results known for the signatures of bounded variation path will automatically extend to p -geometric rough paths for all p , and hence will also hold for p' -weakly geometric rough paths for all p' . This tool for proving properties of signatures of weakly geometric rough paths break down in infinite dimensions. It is not known whether, in infinite dimensions, there exists weakly geometric rough paths that are not p -geometric rough paths for any p . As a result, to complete the proofs for the main result in [1], it becomes necessary to check that the certain properties of signatures of weakly geometric rough paths still holds in infinite dimensions. In the time between writing the first draft of this article and posting this article, the article [2] has appeared which came up with the idea of using a Lemma similar to Lemma 1.7 to extend algebraic identity in finite dimension to infinite dimensions. For the convenience of the reader, we will leave the proof of Lemma 1.7 and its application in this paper.

Notations and main results

For vector spaces A and B , we shall use $A \otimes_a B$ to denote the algebraic tensor product of A and B , namely,

$$A \otimes_a B = \text{span} \{a \otimes b : a \in A, b \in B\}.$$

Let V be a Banach space. We assume that each $V^{\otimes_a k}$ is equipped with a norm $\|\cdot\|_{V^{\otimes_a k}}$ so that:

1. The family $\{\|\cdot\|_{V^{\otimes_a k}} : k \geq 0\}$ satisfies

$$\|a \otimes b\|_{V^{\otimes_a(m+n)}} \leq \|a\|_{V^{\otimes_a m}} \cdot \|b\|_{V^{\otimes_a n}} \quad (1.2)$$

for $a \in V^{\otimes_a m}$ and $b \in V^{\otimes_a n}$.

2. Let $n \in \mathbb{N}$. For any permutation σ on $\{1, \dots, n\}$ and $v_1, \dots, v_n \in V$,

$$\|v_1 \otimes \dots \otimes v_n\|_{V^{\otimes_a n}} = \|v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(n)}\|_{V^{\otimes_a n}}.$$

For $m, n \in \mathbb{N}$, if we define $V^{\otimes m} \otimes V^{\otimes n}$ to be the completion of $V^{\otimes m} \otimes_a V^{\otimes n}$ under the norm $\|\cdot\|_{V^{\otimes_a(m+n)}}$, we have

$$V^{\otimes_a(m+n)} \subset V^{\otimes m} \otimes_a V^{\otimes n} \subset V^{\otimes(m+n)}.$$

It follows that

$$V^{\otimes m} \otimes V^{\otimes n} \simeq V^{\otimes(m+n)},$$

where \simeq denotes the isomorphism as Banach spaces. Let $\tilde{T}^{(n)}(V)$ denote the truncated formal series of tensors whose scalar component is 1, that is

$$\tilde{T}^{(n)}(V) = 1 \oplus V \oplus V^{\otimes 2} \oplus \dots \oplus V^{\otimes n}.$$

Let $\tilde{T}(\bar{V})$ be the space of formal series of tensors, which are sequences $(a_j)_{j=0}^\infty$ with $a_0 = 1$ and $a_j \in V^{\otimes j}$ equipped with the addition and multiplication

$$\begin{aligned}(a+b)_i &= a_i + b_i; \\ (a \otimes b)_i &= \sum_{i=0}^k a_i \otimes b_{k-i}.\end{aligned}$$

We will use $\pi^{(n)}$ and π_n to denote the projection maps from the formal series of tensors $\tilde{T}(\bar{V})$ onto the truncated tensors $\tilde{T}^{(n)}(V)$ and $V^{\otimes n}$ respectively.

We will define the space $L_a^n(V)$ of Lie polynomials over V of degree n inductively so that if $[x, y] = x \otimes y - y \otimes x$, then

$$\begin{aligned}L_a^1(V) &= V \\ L_a^{n+1}(V) &= \text{Span} \{[x, y] : x \in L_a^n(V), y \in V\}.\end{aligned}$$

We will let

$$\mathcal{L}_a^n(V) = \bigoplus_{i=1}^n L_a^i(V)$$

The exponential map on $\tilde{T}^{(n)}(V)$ is defined by

$$\exp[l] = \pi^{(n)} \left[\sum_{j=0}^n \frac{l^{\otimes j}}{j!} \right],$$

with the convention that $l^{\otimes 0} := \mathbf{1}$. Define a function $\|\cdot\| : \tilde{T}^{(n)}(V) \rightarrow \mathbb{R}$ by

$$\|\cdot\|_{\tilde{T}^{(n)}(V)} : = \max_{1 \leq k \leq n} \|\pi_k(\cdot)\|_{V^{\otimes k}}^{\frac{1}{k}}. \quad (1.3)$$

We will let $G^{(n)}$ denote $\overline{\exp \mathcal{L}_a^{[p]}(V)}$.

Definition 1.1. We say a path $x : [0, T] \rightarrow \tilde{T}^{(n)}(V)$ has finite p -variation if

$$\sup_{\mathcal{P}} \left(\sum_{t_j \in \mathcal{P}} \|x_{t_j}^{-1} x_{t_{j+1}}\|_{\tilde{T}^{(n)}(V)}^p \right)^{\frac{1}{p}} < \infty.$$

Let $p \geq 1$. We say a path $x : [0, T] \rightarrow \tilde{T}^{(\lfloor p \rfloor)}(V)$ is a p -weakly geometric rough path if $x_t \in G^{(\lfloor p \rfloor)}$ and x has finite p -variation.

We are mainly interested in the following special transform of weakly geometric rough paths.

Definition 1.2. (Lyons, Theorem 2.2.1 [9]) For all p -weakly geometric rough paths $x : [0, T] \rightarrow \tilde{T}^{(\lfloor p \rfloor)}(V)$ and all $n \geq \lfloor p \rfloor$, there exists a unique $S_n(x)_{0,\cdot} : [0, T] \rightarrow \tilde{T}^{(n)}(V)$ such that $S_n(x)_{0,0} = \mathbf{1}$, S_n has finite p -variation and

$$\pi^{(\lfloor p \rfloor)} \left(S_n(x)_{0,t} \right) = x_0^{-1} x_t.$$

for all t . The unique element $S(x)_{0,T} \in \tilde{T}((V))$ such that $\pi^{(n)} \left(S(x)_{0,T} \right) = S_n(x)_{0,T}$ for all n is called the signature of the path x .

Weakly geometric rough path in Banach space

The results we will prove in this section are:

Lemma 1.1. *The function $\|\cdot\|_{\tilde{T}^{(n)}(V)}$, defined in (1.3), is symmetric on $G^{(n)}$ in the sense that for all $g \in G^{(n)}$,*

$$\|g\|_{\tilde{T}^{(n)}(V)} = \|g^{-1}\|_{\tilde{T}^{(n)}(V)}.$$

Lemma 1.2. *Let $N \in \mathbb{N}$. There exists a map $\mathcal{J} : WG\Omega_p(V) \rightarrow WG\Omega_p \left(\bigoplus_{i=0}^N V^{\otimes i} \right)$ such that for all $x \in WG\Omega_p(V)$:*

1. $\pi_1(\mathcal{J}(x)) = S_N(x)_{0,\cdot}$;
2. *If $x, y \in WG\Omega_p(V)$ is such that $S(x) = S(y)$, then $S(\mathcal{J}(x)) = S(\mathcal{J}(y))$.*

Lemma 1.3. *Let W be a Banach space and $\Phi : W \rightarrow \mathbb{R}^d$ be a continuous linear functional on W . Then there exists a map $\mathbf{F} : WG\Omega_p(W) \rightarrow WG\Omega_p(\mathbb{R}^d)$ such that for all $x \in WG\Omega_p(W)$:*

1. $\pi_1(\mathbf{F}(x)) = \Phi(\pi_1(x))$;
2. *If $x, y \in WG\Omega_p(W)$ is such that $S(x) = S(y)$, then $S(\mathbf{F}(x)) = S(\mathbf{F}(y))$.*

Let $x : [0, T] \rightarrow G^{(\lfloor p \rfloor)}$ be a weakly geometric rough path. Define $\overleftarrow{x} : [0, T] \rightarrow G^{(\lfloor p \rfloor)}$ by

$$\overleftarrow{x}_t = x_{T-t}.$$

Lemma 1.4. *Let x be a weakly geometric rough path. Then $S(\overleftarrow{x})_{0,T} = S(x)_{0,T}^{-1}$.*

Lemma 1.5. *Let $x \in WG\Omega_p(V)$. Let σ be a continuous non-decreasing function. Then $S(x \circ \sigma) = S(x)$.*

Let $\tilde{T}_{i.r.c.}((V))$ (i.r.c. for infinite radius of convergence) be the set of elements in $\tilde{T}((V))$ such that

$$\|\cdot\| := \max_{k \in \mathbb{N}} \|\pi_k(\cdot)\|_{V^{\otimes k}}^{\frac{1}{k}} < \infty.$$

By Lemma 1.1, $\|a\| = \|a^{-1}\|$ and if $a, b \in \tilde{T}_{i.r.c.}((V))$, then $a \otimes b \in \tilde{T}_{i.r.c.}((V))$ and $\|a + b\| \leq \|a\| + \|b\|$. We define a metric d on $\tilde{T}_{i.r.c.}((V))$ by

$$d(a, b) = \|a^{-1} \otimes b\|.$$

Subsequently, when there is no confusion, we will use the shorthand ab to denote $a \otimes b$.

Lemma 1.6. *Let $[x(n)]_{n=0}^{\infty}$ be a sequence in $WG\Omega_p(V)$ such that $\sup_n \|x(n)\|_{p-var} < \infty$ and $[x(n)]_{n=0}^{\infty}$ converges uniformly to x . Then $[x(n)]_{n=0}^{\infty}$ has a subsequence $[x(n_k)]_{k=0}^{\infty}$ such that $S(x(n_k)) \rightarrow S(x)$.*

Symmetric norm

The following Lemma useful as finite dimensional projection type of result and has first appeared in [2].

Lemma 1.7. *Let V be a Banach space and $M \in \mathbb{N}$. Let $X \in \exp \mathcal{L}_a^M(V)$. Then there exists a finite dimensional subspace V' of V such that $X \in \exp \mathcal{L}_a^M(V')$.*

Proof. Since $X \in \bigoplus_{k=0}^M V^{\otimes_a k}$, there exists a finite set $J \subset \mathbb{N}$, $(X_{ji}^k)_{1 \leq i \leq k, j \in J}$, $X_{ji}^k \in V$ such that

$$X = \mathbf{1} + \sum_{k=1}^M \sum_{j \in J} X_{j1}^k \otimes \dots \otimes X_{jk}^k.$$

Define $\log : \tilde{T}^{(n)}(V) \rightarrow \tilde{T}^{(n)}(V)$ by

$$\log(1 + g) = \pi^{(n)} \left(\sum_{k=1}^n (-1)^{k+1} \frac{g^k}{k} \right). \quad (1.4)$$

There exists a finite set $J' \subset \mathbb{N}$, $(l_{ji}^k)_{1 \leq i \leq k, l_{ji}^k \in V}$, such that

$$\log X = \sum_{k=1}^M \sum_{j \in J'} \left[l_{j1}^k, \left[\dots \left[l_{j(k-1)}^k, l_{jk}^k \right] \right] \right],$$

where $[c, d] = c \otimes d - d \otimes c$. Let

$$V' := \text{Span} \{ l_{ji}^k : 1 \leq i \leq k, 1 \leq k \leq M, j \in J' \}.$$

In particular, X lies in $\tilde{T}^{(M)}(V')$ and $\log X \in \mathcal{L}_a^M(V')$. \square

Proof of Lemma 1.1. As the map $g \rightarrow g^{-1}$ is continuous with respect to the $\|\cdot\|_{\tilde{T}^{(n)}(V)}$, it suffices to prove the Lemma for elements in $\exp \mathcal{L}_a^n(V)$. Let $g \in \exp \mathcal{L}_a^n(V)$. By Lemma 1.7, there exists a finite dimensional subspace V' of V such that $g \in \exp \mathcal{L}_a^n(V')$. Define the antipode operator $\alpha : G^{(*)} \rightarrow G^{(*)}$ (see also p13, [3]) so that

$$\pi_k(\alpha(v_1 \otimes \dots \otimes v_k)) = (-1)^k v_k \otimes \dots \otimes v_1.$$

It follows from Theorem 3.3.3 in [10] that

$$\pi_n(g^{-1}) = \pi_n(\alpha(g)). \quad (1.5)$$

Therefore, in particular, we have that $\|\cdot\|_{\tilde{T}^{(n)}(V)}$ is symmetric. \square

Transformation on rough paths

Signature of signature

A key property of iterated integrals not just for our purpose but also for rough path theory in general, is that the iterated integrals of iterated integrals can be rewritten as a linear combination of iterated integrals. We will now make this precise in the infinite dimensional setting.

Fix $N \in \mathbb{N}$, define

$$W = \bigoplus_{i=1}^N V^{\otimes i}.$$

It follows that

$$W^{\otimes a n} = \bigoplus_{i_1, \dots, i_n=1}^N V^{\otimes i_1} \otimes_a \dots \otimes_a V^{\otimes i_n} \subset \bigoplus_{i_1, \dots, i_n=1}^N V^{\otimes (i_1 + \dots + i_n)}.$$

We define the tensor norm $\|\cdot\|_{W^{\otimes n}}$ on $W^{\otimes a n}$ by

$$\|\xi\|_{W^{\otimes n}} = \sum_{i_1, \dots, i_n=1}^N \|\xi^{i_1, \dots, i_n}\|_{V^{\otimes (i_1 + \dots + i_n)}},$$

where $\xi = (\xi^{i_1, \dots, i_n})_{1 \leq i_1, \dots, i_n \leq N}$. Note that by the admissibility of $(\|\cdot\|_{V^{\otimes k}} : k \geq 1)$,

$$\begin{aligned} & \left(\sum_{1 \leq i_1, \dots, i_n \leq N} \|\xi^{i_1, \dots, i_n}\|_{V^{\otimes (i_1 + \dots + i_n)}} \right) \left(\sum_{1 \leq j_1, \dots, j_k \leq N} \|\xi^{j_1, \dots, j_k}\|_{V^{\otimes (j_1 + \dots + j_k)}} \right) \\ &= \sum_{1 \leq i_1, \dots, i_n \leq N, 1 \leq j_1, \dots, j_k \leq N} \|\xi^{i_1, \dots, i_n}\|_{V^{\otimes (i_1 + \dots + i_n)}} \|\xi^{j_1, \dots, j_k}\|_{V^{\otimes (j_1 + \dots + j_k)}} \\ &\geq \sum_{1 \leq i_1, \dots, i_{n+k} \leq N} \|\xi^{i_1, \dots, i_{n+k}}\|_{V^{\otimes (i_1 + \dots + i_{n+k})}}. \end{aligned}$$

By taking completion, $W^{\otimes n}$ coincides with $\bigoplus_{i_1, \dots, i_n=1}^N V^{\otimes (i_1 + \dots + i_n)}$. Let $\overline{\mathcal{L}_a^n(V)}$ be the closure of $\mathcal{L}_a^n(V)$ in $\tilde{T}^{(n)}(V)$. By admissibility (1.2) of the tensor norm $\|\cdot\|_{V^{\otimes m}}$, the multiplication operator $\otimes : (V^{\otimes k}, V^{\otimes k'}) \rightarrow V^{\otimes (k+k')}$ is continuous. This implies that the functions exp and log are both continuous. Hence

$$\overline{\exp(\mathcal{L}_a^n(V))} = \exp(\overline{\mathcal{L}_a^n(V)}).$$

Proof of Lemma 1.2. Define for $w \in V^{\otimes j}$ and $v \in V^{\otimes k}$,

$$F_j(v)(w) = v \otimes w$$

and for $v_1 \in V^{\otimes k_1}, \dots, v_n \in V^{\otimes k_n}$ and $w_1 \in V^{\otimes j_1}, \dots, w_n \in V^{\otimes j_n}$,

$$F_{j_1, \dots, j_n}(v_1, \dots, v_n)(w_1 \otimes \dots \otimes w_n) = F_{j_1}(v_1)(w_1) \otimes \dots \otimes F_{j_n}(v_n)(w_n).$$

Define for a permutation π on $\{1, \dots, n\}$, the operator π on $V^{\otimes n}$ by

$$\pi(v_1 \otimes \dots \otimes v_n) = v_{\pi(1)} \otimes \dots \otimes v_{\pi(n)}.$$

We define for $X, Y \in T^{(nN)}(V)$,

$$H_{i_1, \dots, i_n}(X, Y) = \sum_{j_n=1}^{i_n} \dots \sum_{j_1=1}^{i_1} F_{j_1, \dots, j_n}(X^{i_1-j_1}, \dots, X^{i_n-j_n}) \left[\sum_{\pi \in OS(j_1, \dots, j_n)} \pi(Y^{j_1+\dots+j_n}) \right] \quad (1.6)$$

where for $Z \in T^{(nN)}(V)$, we use the notation Z^k to denote $\pi_k(Z)$ and $OS(j_1, \dots, j_n)$ to denote the set of ordered shuffles (see p72 [11]). We define $\mathcal{J}(x)$ so that

$$S(\mathcal{J}(x))_{s,t} = H_{i_1, \dots, i_n}(S_{nN}(x)_{0,s}, S_{nN}(x)_{s,t}).$$

As each j_1, \dots, j_n in the sum in (1.6) are at least 1, $S(\mathcal{J}(x))$ has finite p -variation for $x \in WG\Omega_p(V)$. It remains to show that $S(\mathcal{J}(x))$ is multiplicative and takes value in the space of group-like elements. In the finite dimension case, we have proved these properties of $S(\mathcal{J}(x))$ as lemma 4.4 in [1]. We will now use Lemma 1.7 to carry out a finite dimensional approximation. Let $(s, u, t) \in [0, 1]^3$ be such that $s \leq u \leq t$. Let $1 \leq n \leq [p]$. By Corollary 3.9 in [2], $\pi^{(nN)}(S(x)_{0,s})$, $\pi^{(nN)}(S(x)_{s,u})$ and $\pi^{(nN)}(S(x)_{u,t})$ all lie in $G^{(nN)}$. Therefore, there exist sequences $x^r, y^r, z^r \in \exp \mathcal{L}_a^{nN}(V)$ such that $x^r \rightarrow \pi^{(nN)}(S(x)_{0,s})$, $y^r \rightarrow \pi^{(nN)}(S(x)_{s,u})$ and $z^r \rightarrow \pi^{(nN)}(S(x)_{u,t})$ as $r \rightarrow \infty$. By Lemma 1.7, there exists finite dimensional spaces V_r such that $x_r, y_r, z_r \in \exp \mathcal{L}_a^{nN}(V_r)$. By the Chow-Rashevskii's theorem ([12, 5] or see for example Theorem 7.28 in [8]), there exist bounded variation paths a, b, c in V_r such that $S_{nN}(a)_{0,s} = x^r$, $S_{nN}(b)_{s,u} = y^r$ and $S_{nN}(c)_{u,t} = z^r$. Let ξ denote the path $a \star b \star c$, where \star denotes the concatenation of paths. Then for $i_1 + \dots + i_n \leq nN$,

$$H_{i_1, \dots, i_n}(x^r, y^r z^r) = H_{i_1, \dots, i_n}(S_{nN}(\xi)_{0,s}, S_{nN}(\xi)_{s,t})$$

and the same holds when s, x^r and $y^r z^r$ are replaced, respectively by $u, x^r y^r$ and z^r .

By the computation leading to (4.4) in [1], for $i_1 + \dots + i_n \leq nN$,

$$\begin{aligned} & \int_{s < s_1 < \dots < s_n < t} dS_{nN}(\xi)^{i_1}_{0,s_1} \otimes \dots \otimes dS_{nN}(\xi)^{i_n}_{0,s_n} \\ &= H_{i_1, \dots, i_n}(S_{nN}(\xi)_{0,s}, S_{nN}(\xi)_{s,t}). \end{aligned}$$

The same holds when (s, u) is replaced by (u, t) . As the iterated integrals of the bounded variation path

$$s \rightarrow S_{nN}(\xi)_{0,s}$$

has the multiplicative property, we also have

$$H_{i_1, \dots, i_n}(x^r, y^r z^r) = \sum_{j=0}^n H_{i_1, \dots, i_j}(x^r, y^r) \otimes H_{i_{j+1}, \dots, i_n}(x^r y^r, z^r).$$

Moreover, as the signature of a bounded variation path ξ is a group-like element, for all t ,

$$\sum_{i_n=0}^N \dots \sum_{i_1=0}^N H_{i_1, \dots, i_n}(\mathbf{1}, S_{nN}(\xi)_{0,t}) \in \exp(\mathcal{L}_a^{nN}(V_r)) \subset \exp(\mathcal{L}_a^{nN}(V)).$$

By taking limit as $r \rightarrow \infty$ and using the continuity of the map H , the map H satisfies the multiplicative property

$$\begin{aligned} & \sum_{j=0}^n H_{i_1, \dots, i_j}(S_{nN}(x)_{0,s}, S_{nN}(x)_{s,u}) \otimes H_{i_{j+1}, \dots, i_n}(S_{nN}(x)_{0,u}, S_{nN}(x)_{u,t}) \\ &= H_{i_1, \dots, i_n}(S_{nN}(x)_{0,s}, S_{nN}(x)_{s,t}). \end{aligned}$$

and lies in the space of group-like elements

$$\sum_{i_n=0}^N \dots \sum_{i_1=0}^N H_{i_1, \dots, i_n}(\mathbf{1}, S_{nN}(x)_{0,t}) \in \overline{\exp(\mathcal{L}_a^{nN}(V))}.$$

This holds for all s, u and t and hence Z is a weakly geometric rough path. \square

Linear map on rough paths

Proof of Lemma 1.3. For any linear map $\Phi : W \rightarrow \mathbb{R}^d$, we may continuously extend Φ to a linear operator on $T^{(N)}(W)$ such that for $w_1, \dots, w_N \in W$,

$$\Phi(w_1 \otimes \dots \otimes w_N) = \Phi(w_1) \otimes \dots \otimes \Phi(w_N).$$

Let $x \in WG\Omega_p(W)$. As Φ is a bounded linear operator and the norm on $T^{(N)}(W)$ is admissible, $\Phi(x)$ has finite p -variation. As Φ is a homomorphism with respect to \otimes , for all $t \geq 0$, $\Phi(x_t)$ lies in the $[p]$ -step free nilpotent Lie group over \mathbb{R}^d . Therefore, $\Phi(x) \in WG\Omega_p(\mathbb{R}^d)$. By construction, $\pi_1(\Phi(x)) = \Phi(\pi_1(x))$. Moreover, again by the homomorphism property of Φ and admissibility of the norm on $T^{(N)}(W)$, we have

$$\Phi(S_N(x)) = S_N(\Phi(x))$$

which implies property 2. in the Lemma. \square

Signature of the reversed path

Let x be a weakly geometric rough path on an interval $[0, T]$. We recall that the reversal of x , denoted as \overleftarrow{x} , is defined so that

$$\overleftarrow{x}_t = x_{T-t}.$$

We will now prove Lemma 1.4 that

$$S(x)_{0,T} \otimes S(\overleftarrow{x})_{0,T} = \mathbf{1}.$$

Proof of Lemma 1.4. We will prove a stronger fact that for all $s \leq t$ and all n ,

$$S_n(x)_{s,t} \otimes S_n(\overleftarrow{x})_{T-t,T-s} = \mathbf{1}$$

by induction on n , where $S_n(x) = \pi^{(n)}(S(x))$ and $S(x)_{s,t} = S(x|_{[s,t]})$. The base induction case of $n = \lfloor p \rfloor$ is obvious since $S_{\lfloor p \rfloor}(\overleftarrow{x})_{T-t,T-s}$ is by definition equals to $\overleftarrow{x}_{T-t}^{-1} \overleftarrow{x}_{T-s} = x_t^{-1} x_s$. We will use the notation \mathbb{X}^i and $\hat{\mathbb{X}}^i$ to denote, respectively, $\pi_i(S(x))$ and $\pi_i(S(\overleftarrow{x}))$. By (2.2.9) in [9],

$$\begin{aligned} & S_{n+1}(x)_{s,t} \otimes S_{n+1}(\overleftarrow{x})_{T-t,T-s} \\ &= \lim_{\max_i |u_{i+1} - u_i| \rightarrow 0} \hat{\mathbb{X}}_{u_0, u_1} \otimes \dots \otimes \hat{\mathbb{X}}_{u_{l-1}, u_l} \otimes \lim_{\max |s_{i+1} - s_i| \rightarrow 0} \hat{\mathbb{X}}_{s_0, s_1} \otimes \dots \otimes \hat{\mathbb{X}}_{s_{k-1}, s_k}. \end{aligned}$$

where $\hat{\mathbb{X}}_{s,t}$ and $\hat{\mathbb{X}}_{s,t}$ denote respectively elements $(S_n(x)_{s,t}, 0)$ and $(S_n(\overleftarrow{x})_{s,t}, 0)$ in $T^{(n+1)}(V)$, $s = u_0 < u_1 < \dots < u_l = t$ and $T-t = s_0 < s_1 < \dots < s_k = T-s$. In particular, we have

$$\begin{aligned} & S_{n+1}(x)_{s,t} \otimes S_{n+1}(\overleftarrow{x})_{T-t,T-s} \\ &= \lim_{\max |u_{i+1} - u_i| \rightarrow 0} \hat{\mathbb{X}}_{u_0, u_1} \otimes \dots \otimes \hat{\mathbb{X}}_{u_{l-1}, u_l} \otimes \hat{\mathbb{X}}_{T-u_l, T-u_{l-1}} \otimes \dots \otimes \hat{\mathbb{X}}_{T-u_1, T-u_0} \end{aligned}$$

where the tensor product \otimes is taken in $T^{(n+1)}(V)$. We claim that for all $t_0 < \dots < t_l$,

$$\begin{aligned} & \hat{\mathbb{X}}_{t_0, t_1} \otimes \dots \otimes \hat{\mathbb{X}}_{t_{l-1}, t_l} \otimes \hat{\mathbb{X}}_{T-t_l, T-t_{l-1}} \otimes \dots \otimes \hat{\mathbb{X}}_{T-t_1, T-t_0} \\ &= \left(1, 0, \dots, 0, \sum_{j=0}^{l-1} \sum_{k=1}^n \mathbb{X}_{t_j, t_{j+1}}^k \otimes \overleftarrow{\mathbb{X}}_{T-t_{j+1}, T-t_j}^{n+1-k} \right) \end{aligned}$$

by induction on l . For the base induction step, note that by the induction hypothesis over n ,

$$\begin{aligned} \pi^{(n)} \left(\hat{\mathbb{X}}_{t_0, t_1} \otimes \hat{\mathbb{X}}_{T-t_1, T-t_0} \right) &= S_n(x)_{t_0, t_1} \otimes S_n(\overleftarrow{x})_{T-t_1, T-t_0} \\ &= (1, 0, \dots, 0) \end{aligned}$$

and

$$\pi_{n+1} \left(\hat{\mathbb{X}}_{t_0, t_1} \otimes \hat{\mathbb{X}}_{T-t_1, T-t_0} \right) = \sum_{k=1}^n \mathbb{X}_{t_0, t_1}^k \otimes \overleftarrow{\mathbb{X}}_{T-t_1, T-t_0}^{n+1-k}.$$

For the induction step, we have by induction hypothesis that

$$\begin{aligned} & \hat{\mathbb{X}}_{t_1, t_2} \otimes \dots \otimes \hat{\mathbb{X}}_{t_{l-1}, t_l} \otimes \hat{\mathbb{X}}_{T-t_l, T-t_{l-1}} \otimes \dots \otimes \hat{\mathbb{X}}_{T-t_2, T-t_1} \\ &= \left(1, 0, \dots, 0, \sum_{j=1}^{l-1} \sum_{k=1}^n \mathbb{X}_{t_j, t_{j+1}}^k \otimes \overleftarrow{\mathbb{X}}_{T-t_{j+1}, T-t_j}^{n+1-k} \right). \end{aligned}$$

To conclude the proof for the claim, it suffices to observe that

$$\begin{aligned} & \left(S_n(x)_{t_0, t_1}, 0 \right) \otimes \left(1, 0, \dots, 0, \sum_{j=1}^{l-1} \sum_{k=1}^n \mathbb{X}_{t_j, t_{j+1}}^k \otimes \overleftarrow{\mathbb{X}}_{T-t_{j+1}, T-t_j}^{n+1-k} \right) \otimes \left(S_n(\overleftarrow{x})_{T-t_1, T-t_0}, 0 \right) \\ &= \left(S_n(x)_{t_0, t_1}, \sum_{j=1}^{l-1} \sum_{k=1}^n \mathbb{X}_{t_j, t_{j+1}}^k \otimes \overleftarrow{\mathbb{X}}_{T-t_{j+1}, T-t_j}^{n+1-k} \right) \otimes \left(S_n(\overleftarrow{x})_{T-t_1, T-t_0}, 0 \right) \\ &= \left(1, 0, \dots, 0, \sum_{j=0}^{l-1} \sum_{k=1}^n \mathbb{X}_{t_j, t_{j+1}}^k \otimes \overleftarrow{\mathbb{X}}_{T-t_{j+1}, T-t_j}^{n+1-k} \right). \end{aligned}$$

We now observe, using the claim, that

$$\begin{aligned} & \left\| \hat{\mathbb{X}}_{t_0, t_1} \otimes \dots \otimes \hat{\mathbb{X}}_{t_{l-1}, t_l} \otimes \hat{\mathbb{X}}_{T-t_l, T-t_{l-1}} \otimes \dots \otimes \hat{\mathbb{X}}_{T-t_1, T-t_0} \right\| \\ &\leq \sum_{j=0}^{l-1} \sum_{k=1}^n \left\| \mathbb{X}_{t_j, t_{j+1}}^k \otimes \overleftarrow{\mathbb{X}}_{T-t_{j+1}, T-t_j}^{n+1-k} \right\| \\ &\leq \sum_{j=0}^{l-1} \sum_{k=1}^n \left\| \mathbb{X}_{t_j, t_{j+1}}^k \right\| \left\| \overleftarrow{\mathbb{X}}_{T-t_{j+1}, T-t_j}^{n+1-k} \right\|. \end{aligned} \tag{1.7}$$

Note that $\overleftarrow{\mathbb{X}}_{T-t_{j+1}, T-t_j}^{n+1-k} = \pi_{n+1-k} \left(S_{n+1-k}(x)_{t_j, t_{j+1}}^{-1} \right)$ by induction hypothesis. By (1.5) in the proof of Lemma 1.1,

$$\left\| \overleftarrow{\mathbb{X}}_{T-t_{j+1}, T-t_j}^{n+1-k} \right\| = \left\| \mathbb{X}_{t_j, t_{j+1}}^{n+1-k} \right\|.$$

Therefore, by (1.7),

$$\begin{aligned} & \left\| \hat{\mathbb{X}}_{t_0, t_1} \otimes \dots \otimes \hat{\mathbb{X}}_{t_{l-1}, t_l} \otimes \hat{\mathbb{X}}_{T-t_l, T-t_{l-1}} \otimes \dots \otimes \hat{\mathbb{X}}_{T-t_1, T-t_0} \right\| \\ &\leq \sum_{j=0}^{l-1} \sum_{k=1}^n \left\| x|_{[t_j, t_{j+1}]} \right\|_{p-var}^k \left\| x|_{[t_j, t_{j+1}]} \right\|_{p-var}^{n+1-k} \\ &\leq n \max_j \left\| x|_{[t_j, t_{j+1}]} \right\|_{p-var}^{n+1-p} \left\| x|_{[s, t]} \right\|_{p-var}^p \\ &\rightarrow 0 \end{aligned}$$

as $\max_j |t_{j+1} - t_j| \rightarrow 0$.

□

Invariance of signature under reparametrisation

Proof of Lemma 1.5. We shall prove it by induction on n that

$$S_n(x \circ \sigma)_{0,T} = S_n(x)_{0,T}.$$

We will prove the stronger fact that $S_n(x \circ \sigma)_{s,t} = S_n(x)_{\sigma(s),\sigma(t)}$ for all $s, t \in [0, T]$. Let $S_n(x \circ \sigma)_{s,t} = (1, \mathbb{Y}_{s,t}^1, \dots, \mathbb{Y}_{s,t}^n)$.

Note that

$$\begin{aligned} \mathbb{Y}_{s,t}^{n+1} &= \lim_{|\mathcal{P}| \rightarrow 0} \sum_{j=0}^{l-1} \sum_{k=1}^n \mathbb{Y}_{s,t_j}^k \otimes \mathbb{Y}_{t_j,t_{j+1}}^{n+1-k} \\ &= \lim_{|\mathcal{P}| \rightarrow 0} \sum_{j=0}^{l-1} \sum_{k=1}^n \mathbb{X}_{\sigma(s),\sigma(t_j)}^k \otimes \mathbb{X}_{\sigma(t_j),\sigma(t_{j+1})}^{n+1-k}, \end{aligned}$$

where the second line follows from the induction hypothesis. Let (s_j) be a sequence defined by $s_0 = s$ and

$$s_{j+1} = \min_k \{t_k : \sigma(t_k) > \sigma(s_j)\}.$$

Then as by definition $\mathbb{X}_{u,u}^m = 0$ for any m and any u ,

$$\mathbb{Y}_{s,t}^{n+1} = \lim_{|\mathcal{P}| \rightarrow 0} \sum_j \sum_{k=1}^n \mathbb{X}_{\sigma(s),\sigma(s_j)}^k \otimes \mathbb{X}_{\sigma(s_j),\sigma(s_{j+1})}^{n+1-k}. \quad (1.8)$$

Note that as σ is non-decreasing, the set $(\sigma(s_j) : j \geq 0)$ is a partition of $[\sigma(s), \sigma(t)]$. Moreover, since for all j , there exists m such that $(s_j, s_{j+1}) = (\sigma(t_m), \sigma(t_{m+1}))$ and σ is uniformly continuous, we have that $\max_j |s_{j+1} - s_j| \rightarrow 0$ as $\max_m |t_{m+1} - t_m| \rightarrow 0$. Therefore, by (1.8), $\mathbb{Y}_{s,t}^{n+1} = \mathbb{X}_{\sigma(s),\sigma(t)}^{n+1}$.

□

1.1 Uniform convergence + Bounded in p -variation \implies Subsequential convergence of Signatures

The following proof is simply a matter of checking that the proof in [8] applies in the infinite dimensional setting.

Proof of Lemma 1.6. By Theorem 2.2.2 in [9], it suffices to show that there exists p' and a control ω (see Definition 1.9 in [11]) such that for all $1 \leq i \leq \lfloor p \rfloor$ and $s \leq t$,

$$\left\| \pi_i(x(n_k)_{s,t}) \right\| \leq \omega(s, t)^{\frac{i}{p'}}, \quad \|\pi_i(x_{s,t})\| \leq \omega(s, t)^{\frac{i}{p'}}$$

and a sequence $a(n_k) \rightarrow 0$ such that

$$\left\| \pi_i \left(x(n_k)_{s,t} - x_{s,t} \right) \right\| \leq a(n_k) \omega(s, t)^{\frac{i}{p'}}.$$

By Theorem 3.3.3 in [9], it suffices to show that for some $p' \geq 1$,

$$d_{p'-var}(x(n), x) \rightarrow 0$$

as $n \rightarrow \infty$, where $d_{p'-var}$ is defined by (1.9) below.

Take $p' > p$ such that $\lfloor p \rfloor = \lfloor p' \rfloor$. For $x \in WG\Omega_p(V)$, let $x_{s,t}$ denote $x_s^{-1}x_t$. Then

$$d_{p'-var}(x, y) = \max_{1 \leq i \leq \lfloor p' \rfloor} \left(\sum_j \left\| \pi_i(x_{t_j, t_{j+1}} - y_{t_j, t_{j+1}}) \right\|^{\frac{p'}{i}} \right)^{\frac{i}{p'}} \quad (1.9)$$

$$\begin{aligned} &\leq \sup_{s \leq t} \|x_{s,t} - y_{s,t}\|^{\frac{p'-p}{p'}} \max_{1 \leq i \leq \lfloor p' \rfloor} \left(\sum_j \left\| \pi_i(x_{t_j, t_{j+1}} - y_{t_j, t_{j+1}}) \right\|^{\frac{p'}{i}} \right)^{\frac{i}{p'}} \\ &\leq \sup_{s \leq t} \|x_{s,t} - y_{s,t}\|^{\frac{p'-p}{p'}} 2^{p-1} \max_{1 \leq i \leq \lfloor p' \rfloor} \left(\sum_j \left\| \pi_i(x_{t_j, t_{j+1}}) \right\|^{\frac{p'}{i}} + \left\| \pi_i(y_{t_j, t_{j+1}}) \right\|^{\frac{p'}{i}} \right)^{\frac{i}{p'}} \end{aligned} \quad (1.10)$$

$$\begin{aligned} &\leq \sup_{s \leq t} \|x_{s,t} - y_{s,t}\|^{\frac{p'-p}{p'}} 2^{p-1} \left[\max_{1 \leq i \leq \lfloor p' \rfloor} \left(\sum_j \left\| \pi_i(x_{t_j, t_{j+1}}) \right\|^{\frac{p'}{i}} \right)^{\frac{i}{p'}} \right. \\ &\quad \left. + \max_{1 \leq i \leq \lfloor p' \rfloor} \left(\sum_j \left\| \pi_i(y_{t_j, t_{j+1}}) \right\|^{\frac{p'}{i}} \right)^{\frac{i}{p'}} \right]. \end{aligned} \quad (1.11)$$

Note that

$$\begin{aligned} \left\| \pi_i(x_{s,t} - y_{s,t}) \right\| &\leq \left\| \pi_i((x_s^{-1} - y_s^{-1})x_t + y_s^{-1}(x_t - y_t)) \right\| \\ &\leq \sum_{j=0}^i \|x_s^{-1} - y_s^{-1}\|^j \|x_t\|^{i-j} + \|x_s - y_s\|^j \|y_s^{-1}\|^{i-j} \end{aligned} \quad (1.12)$$

If $y = x(n)$ and $x(n) \rightarrow x$ uniformly, then since we have for $g = (1, g^1, \dots, g^{\lfloor p \rfloor})$,

$$\begin{aligned} g^{-1} &= \sum_{j=0}^{\lfloor p \rfloor} (-1)^j (g-1)^{\otimes j} \\ &= \sum_{j=0}^{\lfloor p \rfloor} (-1)^j \sum_{i_1, \dots, i_j \geq 1} g^{i_1} \dots g^{i_j} \end{aligned}$$

and the tensor product is continuous,

$$\lim_{n \rightarrow \infty} \sup_s \left\| x_s^{-1} - x(n)_s^{-1} \right\| = 0.$$

By (1.12), $\sup_{s \leq t} \|x_{s,t} - x(n)_{s,t}\| \rightarrow 0$ as $n \rightarrow \infty$ and by (1.10), $d_{p'-var}(x(n), x) \rightarrow 0$ as $n \rightarrow \infty$. □

Partially ordered sets as \mathbb{R} -trees

Finally, we prove the following lemma which characterise partially ordered sets that can be realised as \mathbb{R} -trees.

Lemma 1.8. *Let (\mathcal{P}, \preceq) be a partially ordered set such that:*

1. \mathcal{P} has a least element v .
2. For all $\tau_1 \in \mathcal{P}$, the set $\{\tau_2 \in \mathcal{P} : \tau_2 \preceq \tau_1\}$ is totally ordered.
3. For any $\tau_1, \tau_2 \in \mathcal{P}$, the set $\{\tau_3 \in \mathcal{P} : \tau_3 \preceq \tau_1, \tau_3 \preceq \tau_2\}$ has a unique maximal element, which will be denoted as $\tau_1 \wedge \tau_2$.
4. There exists a function $\alpha : \mathcal{P} \rightarrow [0, \infty)$ such that $\alpha(v) = 0$ and the restriction of α on the set $\{\tau_3 \in \mathcal{P} : \tau_3 \preceq \tau_1\}$ is strictly increasing for any $\tau_1 \in \mathcal{P}$.
then

(i) for all $\tau_1, \tau_2, \tau_3 \in \mathcal{P}$,

$$\alpha(\tau_1 \wedge \tau_2) \geq \min(\alpha(\tau_1 \wedge \tau_3), \alpha(\tau_2 \wedge \tau_3)); \quad (1.13)$$

(ii) if we define

$$d(\tau_1, \tau_2) = \alpha(\tau_1) + \alpha(\tau_2) - 2\alpha(\tau_1 \wedge \tau_2),$$

then (\mathcal{P}, d) is a metric space that is 0-hyperbolic with respect to v in the sense of [4] (p11.).

(iii) (\mathcal{P}, d) is a \mathbb{R} -tree.

Remark 1.1. This Lemma is essentially Proposition 3.10 in [6]. Here we provide an alternative proof.

Proof. (i) As $\{\tau_4 : \tau_4 \preceq \tau_3\}$ is totally ordered, we may assume without loss of generality that

$$\tau_1 \wedge \tau_3 \preceq \tau_2 \wedge \tau_3.$$

This implies in particular that $\tau_1 \wedge \tau_3 \preceq \tau_2$. However, by the definition of \wedge , we also have $\tau_1 \wedge \tau_3 \preceq \tau_1$. Therefore, $\tau_1 \wedge \tau_3 \preceq \tau_1 \wedge \tau_2$. The inequality (1.13) now follows from the assumption 4. in the Lemma.

(ii) The only thing that needs proving is the triangle inequality for the metric d and the property of 0-hyperbolic. Both of which are equivalent to (1.13).

(iii) By Lemma 4.13 in [4], a metric space is an \mathbb{R} -tree if and only if it is connected and 0-hyperbolic. The metric space (\mathcal{P}, d) is path-connected by Condition 4., and is 0-hyperbolic by (ii), and hence (\mathcal{P}, d) is a \mathbb{R} -tree. □

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References

- [1] H. Boedihardjo, X. Geng, T. Lyons, D. Yang, The signature of a rough path: uniqueness, arXiv: 1406.7871, 2014.
- [2] T. Cass, B. Driver, N. Lim, C. Litterer, On the integration of weakly geometric rough paths, 2014, to appear in the *Journal of Mathematical Society of Japan*.
- [3] I. Chevyrev, T. Lyons, Characteristic functions of measures on geometric rough paths, arXiv: 1307.3508, 2014, to appear in *Annals of Probability*.
- [4] I. Chiswell, *Introduction to Λ -trees*, World Scientific Publishing Co., Inc., River Edge, NJ, 2001.
- [5] W. Chow, Uber System von linearen partiellen Differentialgleichungenerster Ordnung, *Mathematische Annalen*, 117, 98-105, 1939.
- [6] C. Favre, M. Jonsson, *The valuative tree*, Lecture Notes in Math. 1853, Springer, Berlin, 2004.
- [7] P. Friz, N. Victoir, A note on the notion of geometric rough paths, *Probab. Theory Relat. Fields*, 136, 395–416, 2006.
- [8] P. Friz, N. Victoir, *Multidimensional stochastic processes as rough paths*, Cambridge Studies of Advanced Mathematics, Vol. 120, Cambridge University Press, 2010.
- [9] T. Lyons, Differential equations driven by rough signals, *Rev. Mat. Iberoamericana* 14 (2), 215–310, 1998.
- [10] T. Lyons and Z. Qian, *System control and rough paths*, Oxford Mathematical Monographs, Oxford University Press, Oxford, Oxford Science Publications, 2002.
- [11] T. Lyons, M. Caruana, T. Lévy, *Differential equations driven by rough paths*, Springer, 2007.
- [12] P. Rashevskii, About connecting two points of complete non-holonomic space by admissible curve (in Russian), *Uch. Zapiski ped. inst. Libknexa*, 2, 83-94, 1938.